ON QUOTIENT MODULES OF $H^2(\mathbb{D}^n)$: ESSENTIAL NORMALITY AND BOUNDARY REPRESENTATIONS

B. KRISHNA DAS, SUSHIL GORAI, AND JAYDEB SARKAR

ABSTRACT. Let $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$, $n \geq 1$, and let $H^2(\mathbb{D}^n)$ be the Hardy space over $\mathbb{D}^n$. For $n \geq 3$, we show that if $\theta \in H^\infty(\mathbb{D}^n)$ is an inner function, then the n-tuple of commuting operators $(C_{z_1}, \ldots, C_{z_n})$ on the Beurling type quotient module $\mathcal{Q}_\theta$ is not essentially normal, where

$$\mathcal{Q}_\theta = H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n) \quad \text{and} \quad C_{z_j} = P_{\mathcal{Q}_\theta} M_{z_j}|_{\mathcal{Q}_\theta} \quad (j = 1, \ldots, n).$$

Rudin's quotient modules of $H^2(\mathbb{D}^2)$ are also shown to be not essentially normal. We prove several results concerning boundary representations of C^*-algebras corresponding to different classes of quotient modules including doubly commuting quotient modules and homogeneous quotient modules.

1. Introduction

In this paper, we intend to study essential normality and boundary representations of a class of quotient modules of the Hardy module over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $n > 1$. To be more specific, let $H^2(\mathbb{D}^n)$, $n \geq 1$, denote the Hardy space of holomorphic functions on $\mathbb{D}^n$. We also call $H^2(\mathbb{D}^n)$ the Hardy module over $\mathbb{C}[z_1, \ldots, z_n]$ (see Section 2 for definition). Let $(M_{z_1}, \ldots, M_{z_n})$ denote the (commuting) n-tuple of multiplication operators by the coordinate functions on $H^2(\mathbb{D}^n)$. A closed subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is called a *submodule* if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$, and a closed subspace $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ is a *quotient module* if $\mathcal{Q}^\perp$ ($\cong H^2(\mathbb{D}^n)/\mathcal{Q}$) is a submodule. A quotient module $\mathcal{Q}$ is said to be of *Beurling type* [13] if

$$\mathcal{Q} = \mathcal{Q}_\theta := H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n) \cong H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n),$$

for some inner function $\theta \in H^\infty(\mathbb{D}^n)$ (that is, θ is a bounded analytic function on $\mathbb{D}^n$ and $|\theta| = 1$ a.e. on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$). We denote by $\mathcal{S}_\theta$ the submodule $\theta H^2(\mathbb{D}^n)$ of $H^2(\mathbb{D}^n)$. A quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ is *essentially normal* [7] if the commutator $[C_{z_i}, C_{z_j}^*]$ is compact for all $1 \leq i, j \leq n$, where

$$C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} \qquad (i = 1, \ldots, n).$$

Essential normality of Hilbert modules is a much studied object in operator theory and function theory. It establishes important connections between operator theory, algebraic geometry, homology theory and complex analysis through the BDF theory [4]. It is well

2000 *Mathematics Subject Classification.* 47A13, 47A20, 47L25, 47L40, 46L05, 46L06.

Key words and phrases. Hardy space, reproducing kernel Hilbert spaces, quotient modules, essential normality, boundary representation.

known that any proper quotient module of $H^2(\mathbb{D})$ is of Beurling-type and essentially normal. This, however, does not hold in general:

(1) For $n = 2$ a Beurling type quotient module $\mathcal{Q}_\theta \subseteq H^2(\mathbb{D}^2)$ is essentially normal if and only if θ is a rational inner function of degree at most $(1, 1)$ [13].

(2) For $n \geq 2$, a quotient module $\mathcal{Q}$ is a Beurling type quotient module of $H^2(\mathbb{D}^n)$ if and only if $\mathcal{Q}^\perp$ is a doubly commuting submodule [20].

An incomplete list of references on the study of essential normality of different classes of quotient modules, including Clark type quotient modules and homogeneous quotient modules, over the bidisc is: [7], [8], [12], [13], [14] and [21].

In this paper we first investigate the essential normality of certain classes of quotient modules including Beurling-type quotient modules of $H^2(\mathbb{D}^n)$, $n \geq 3$. We prove that the Beurling type quotient modules of $H^2(\mathbb{D}^n)$ $(n \geq 3)$ and Rudin quotient modules of $H^2(\mathbb{D}^2)$ are not essentially normal. We obtain a complete characterization for essential normality of doubly commuting quotient modules of an analytic Hilbert module (defined in Section 2) over $\mathbb{C}[\boldsymbol{z}]$ including $H^2(\mathbb{D}^n)$ and the weighted Bergman modules $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$ $(\boldsymbol{\alpha} \in \mathbb{Z}^n, \alpha_i > -1, i = 1, \ldots, n)$ as special cases $(n \geq 2)$.

We also study boundary representations, in the sense of Arveson ([1], [2]), of the C^*-algebra $C^*(\mathcal{Q})$ for different classes of quotient modules $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$. Here, given a quotient module $\mathcal{Q}$, we denote by $B(\mathcal{Q})$ and $C^*(\mathcal{Q})$ the Banach algebra and the C^*-algebra generated by $\{I_\mathcal{Q}, C_{z_i}\}_{i=1}^n$, respectively. For convenience in notation we put

$$B(\mathcal{Q}) = B(C_{z_1}, \ldots, C_{z_n}), \text{ and } C^*(\mathcal{Q}) = C^*(C_{z_1}, \ldots, C_{z_n}).$$

It is well known that for an essentially normal quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$, $B(\mathcal{Q})$ is an irreducible operator algebra and the C^*-algebra $C^*(\mathcal{Q})$ contains all compact operators on $\mathcal{Q}$ (see Proposition 2.5 in [3], and Theorem 3.3 and Lemma 3.4 in [5]).

Let us also recall the definition of the boundary representations and some relevant results from operator algebras. Let A be an operator algebra with identity, and let $C^*(A)$ be the C^*-algebra generated by A. An irreducible representation ω of $C^*(A)$ is a *boundary representation* relative to A if $\omega|_A$ has a unique completely positive (CP) extension to $C^*(A)$. An operator algebra A has trivial Shilov ideal ([10]) if

$$\bigcap_{\omega \in \mathrm{bdy}(A)} \ker \omega = \{0\},$$

where $\mathrm{bdy}(A)$ denotes the collection of all boundary representations of $C^*(A)$ relative to A. It is of great interest and importance to identify operator algebras with trivial Shilov ideal. In the particular case of irreducible operator algebras containing compact operators, triviality of Shilov ideal and the fact that the identity representation is a boundary representation are closely related.

THEOREM 1.1. ([2, Proposition 2.1.0]) *Let A be an irreducible operator algebra with identity, and let $C^*(A)$ contain all the compact operators. Then A has trivial Shilov ideal if and only if the identity representation of $C^*(A)$ is a boundary representation relative to A.*

In our context, if $\mathcal{Q}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$ then $B(\mathcal{Q})$ is irreducible and $C^*(\mathcal{Q})$ contains all the compact operators on $\mathcal{Q}$. Therefore, it is natural to ask whether the identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$ for the case when $\mathcal{Q}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$. This problem has a complete solution for the case $n = 1$ (see Arveson [1, Theorem 3.5.3],[2, Corollary 1]):

THEOREM 1.2 (Arveson). *Let $\mathcal{Q}_\theta$ be a quotient module of $H^2(\mathbb{D})$. Then the identity representation of $C^*(\mathcal{Q}_\theta)$ is a boundary representation relative to $B(\mathcal{Q}_\theta)$ if and only if Z_θ is a proper subset of $\mathbb{T}$, where Z_θ consists of all points λ on $\mathbb{T}$ for which θ cannot be continued analytically from $\mathbb{D}$ to λ.*

For the class of essentially normal Beurling type quotient modules of $H^2(\mathbb{D}^2)$, the following characterization was obtained in [13].

THEOREM 1.3 (Guo and Wang). *Let $\theta \in H^\infty(\mathbb{D}^2)$ be a rational inner function of degree at most $(1,1)$, and $\mathcal{Q}_\theta$ be the corresponding essentially normal quotient module of $H^2(\mathbb{D}^2)$. Then the identity representation of $C^*(\mathcal{Q}_\theta)$ is a boundary representation relative to $B(\mathcal{Q}_\theta)$ if and only if θ is not a one variable Blaschke factor.*

In this paper, we study the same problem for several classes of quotient modules of some Hilbert modules over $\mathbb{D}^n$, $n \geq 2$. To be more precise, we study boundary representations for doubly commuting quotient modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, and obtain some direct results for the case of $H^2(\mathbb{D}^n)$ and $L^2_a(\mathbb{D}^n)$ $(n \geq 2)$ (see Corollaries 4.3 and 4.4). We also consider the class of homogeneous quotient modules of $H^2(\mathbb{D}^2)$.

The paper is organized as follows. After obtaining some preliminary results in Section 2, we consider essential normality of Beurling type quotient module of $H^2(\mathbb{D}^n)$ $(n \geq 3)$, doubly commuting quotients modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ and Rudin quotient module of $H^2(\mathbb{D}^2)$ in Section 3. Section 4 is devoted to the study of boundary representations for doubly commuting quotient modules. In Section 5, we discuss boundary representations for homogeneous quotient modules of $H^2(\mathbb{D}^2)$.

2. Preparatory results

In this section we recall some definitions, and prove some elementary results which will be used later. We begin by briefly recalling the definition of the Hardy module.

Let $\mathbb{D}^n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| \leq 1, i = 1, \ldots, n\}$ denote the unit polydisc in $\mathbb{C}^n$. We denote by $\mathbb{N}$ the set of all natural numbers including 0. Set $\mathbb{N}^n = \{\boldsymbol{k} = (k_1, \ldots, k_n) : k_j \in \mathbb{N}, j = 1, \ldots, n\}$ and $\boldsymbol{z}^{\boldsymbol{k}} := z_1^{k_1} \cdots z_n^{k_n}$ for all $\boldsymbol{z} \in \mathbb{C}^n$ and $\boldsymbol{k} \in \mathbb{N}^n$. Then the *Hardy space* $H^2(\mathbb{D}^n)$ over the polydisc $\mathbb{D}^n$ is defined as the space of all holomorphic functions $f = \sum_{\boldsymbol{k}\in\mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$ on $\mathbb{D}^n$ such that $\|f\|^2 := \sum_{\boldsymbol{k}\in\mathbb{N}^n} |a_{\boldsymbol{k}}|^2 < \infty$. It is well known that $H^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel

$$\mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} (1 - z_i \bar{w}_i)^{-1}, \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

and $(M_{z_1}, \ldots, M_{z_n})$ is a commuting tuple of isometries on $H^2(\mathbb{D}^n)$, where

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}) \qquad (f \in H^2(\mathbb{D}^n), \boldsymbol{w} \in \mathbb{D}^n, i = 1, \ldots, n).$$

We represent the n-tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$ as a Hilbert module over $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_n]$ with the following module action:

$$\mathbb{C}[\boldsymbol{z}] \times H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n), \quad (p, f) \mapsto p(M_{z_1}, \ldots, M_{z_n})f.$$

With the above module action $H^2(\mathbb{D}^n)$ is called the *Hardy module* over $\mathbb{C}[\boldsymbol{z}]$.

We also need to recall the definition of the normalized kernel function corresponding to the Szegö kernel on $\mathbb{D}^n$. For each $\boldsymbol{w} \in \mathbb{D}^n$, the *normalized kernel* function $K_{\boldsymbol{w}}$ of $H^2(\mathbb{D}^n)$ is defined by

$$K_{\boldsymbol{w}}(\boldsymbol{z}) := \frac{1}{\|\mathbb{S}(\cdot, \boldsymbol{w})\|}\mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} \sqrt{(1-|w_i|^2)}\frac{1}{1-\overline{w}_i z_i} \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

where $\mathbb{S}(\cdot, \boldsymbol{w})(\boldsymbol{z}) = \mathbb{S}(\boldsymbol{z}, \boldsymbol{w})$ for all $\boldsymbol{z} \in \mathbb{D}^n$. This notion is useful when one studies the Hardy space over $\mathbb{D}^n$, $n > 1$.

LEMMA 2.1. *Let $l \in \{1, \ldots, n\}$ be a fixed integer, and let $\boldsymbol{w}_l = (w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_n)$ be a fixed point in $\mathbb{D}^{n-1}$. Then $K_{(\boldsymbol{w}_l, w)}$ converges weakly to 0 as w approaches to $\partial\mathbb{D}$, where $(\boldsymbol{w}_l, w) = (w_1, \ldots, w_{l-1}, w, w_{l+1}, \ldots, w_n)$.*

Proof. For each $p \in \mathbb{C}[\boldsymbol{z}]$,

$$\langle K_{(\boldsymbol{w}_l, w)}, p\rangle = \overline{p(\boldsymbol{w}_l, w)}\sqrt{1-|w|^2} \prod_{i=1, i\neq l}^{n} \sqrt{1-|w_i|^2}, \tag{2.1}$$

which converges to zero as w approaches to $\partial\mathbb{D}$. For an arbitrary $f \in H^2(\mathbb{D}^n)$, the result now follows from the fact that $\|K_{\boldsymbol{\lambda}}\| = 1$ for all $\boldsymbol{\lambda} \in \mathbb{D}^n$ and $\mathbb{C}[\boldsymbol{z}]$ is dense in $H^2(\mathbb{D}^n)$. □

For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$. For an inner function $\theta \in H^\infty(\mathbb{D}^n)$, it is well known that

$$P_{\mathcal{S}_\theta} = M_\theta M_\theta^* \;\text{ and }\; P_{\mathcal{Q}_\theta} = I_{H^2(\mathbb{D}^n)} - M_\theta M_\theta^*,$$

where M_θ is the multiplication operator defined by

$$(M_\theta f)(\boldsymbol{w}) = \theta(\boldsymbol{w})f(\boldsymbol{w}) \qquad (\boldsymbol{w} \in \mathbb{D}^n, f \in H^2(\mathbb{D}^n)).$$

It follows from the reproducing property of the Szegö kernel that

$$M_\theta^* K(\cdot, \boldsymbol{w}) = \overline{\theta(\boldsymbol{w})}K(\cdot, \boldsymbol{w}),$$

where $K(\cdot, \boldsymbol{w}) := K_{\boldsymbol{w}}$, $\boldsymbol{w} \in \mathbb{D}^n$. In particular, one has

$$P_{\mathcal{S}_\theta}(K_{\boldsymbol{w}}) = M_\theta M_\theta^* K_{\boldsymbol{w}} = \overline{\theta(\boldsymbol{w})}\theta K_{\boldsymbol{w}} \qquad (\boldsymbol{w} \in \mathbb{D}^n).$$

These observations yield the following lemma.

LEMMA 2.2. *Let θ be an inner function in $H^\infty(\mathbb{D}^n)$. Then*

$$P_{\mathcal{Q}_\theta}(K_{\boldsymbol{w}}) = (1 - \overline{\theta(\boldsymbol{w})}\theta)K_{\boldsymbol{w}} \qquad (\boldsymbol{w} \in \mathbb{D}^n). \tag{2.2}$$

We now recall the definition of an analytic Hilbert module over $\mathbb{C}[z]$ (see [5]). Let $k : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ be a positive definite function such that $k(z,w)$ is analytic in z and anti-analytic in w. Let $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D},\mathbb{C})$ be the corresponding reproducing kernel Hilbert space, where $\mathcal{O}(\mathbb{D},\mathbb{C})$ denotes the set of all holomorphic functions on the unit disc. The Hilbert space $\mathcal{H}_k$ is said to be a *reproducing kernel Hilbert module* over $\mathbb{C}[z]$ if the multiplication operator M_z is bounded on $\mathcal{H}_k$.

DEFINITION 2.3. A reproducing kernel Hilbert module $\mathcal{H}_k$ over $\mathbb{C}[z]$ is said to be an analytic Hilbert module over $\mathbb{C}[z]$ if $k^{-1}(z,w)$ is a polynomial in z and $\bar{w}$.

Typical examples of analytic Hilbert modules are the Hardy module $H^2(\mathbb{D})$ with Szegö kernel

$$K(z,w) = \frac{1}{1 - z\bar{w}} \quad (z, w \in \mathbb{D})$$

and the weighted Bergman modules $L^2_{a,\alpha}(\mathbb{D})$ $(\alpha > -1, \alpha \in \mathbb{Z})$ with kernel

$$K_{a,\alpha}(z,w) = \frac{1}{(1 - z\bar{w})^{\alpha+2}} \quad (z, w \in \mathbb{D}, \alpha > -1).$$

It is known that a quotient module of an analytic Hilbert module is irreducible, that is, C_z does not have any non-trivial reducing subspace (cf. Theorem 3.3 and Lemma 3.4 in [5]). Using this, we obtain the next lemma.

LEMMA 2.4. *Let $\mathcal{Q}$ be a non-zero quotient module of an analytic Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$. Then $[C_z, C_z^*] = 0$ if and only if $\mathcal{Q}$ is one dimensional.*

Proof. First note that for any non-zero quotient module $\mathcal{Q}$ of $\mathcal{H}$, the C^*-algebra $C^*(\mathcal{Q})$ is irreducible. If C_z is normal, then $C^*(\mathcal{Q}) \subseteq C^*(\mathcal{Q})' = \mathbb{C}I$. Thus $C^*(\mathcal{Q}) = \mathbb{C}I$, and therefore, $\mathcal{Q}$ is one dimensional. The converse part is trivial, and the proof follows. ☐

Let $\{k_i\}_{i=1}^n$ be positive definite functions on $\mathbb{D}\times\mathbb{D}$. Then $\mathcal{H}_K := \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}$ is said to be an *analytic Hilbert module over* $\mathbb{C}[\boldsymbol{z}]$ if $\mathcal{H}_{k_i}$ is an analytic Hilbert module over $\mathbb{C}[z]$ for all $i = 1, \ldots, n$. In this case, $\mathcal{H}_K \subseteq \mathcal{O}(\mathbb{D}^n, \mathbb{C})$ and

$$K(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n k_i(z_i, w_i) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

is the reproducing kernel function of $\mathcal{H}_K$ (cf. [5]). In the sequel, we will often identify M_{z_i} on $\mathcal{H}_K$ with the operator $I_{\mathcal{H}_{k_1}} \otimes \cdots \otimes \underbrace{M_z}_{\text{i-th place}} \otimes \cdots \otimes I_{\mathcal{H}_{k_n}}$, $i = 1, \ldots, n$, on the n-fold Hilbert space tensor product $\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}$. We end this section with a result on essential normality of a Beurling type quotient module $\mathcal{Q}_\theta$, where θ is a one variable inner function in $\mathbb{D}^n$.

LEMMA 2.5. *Let $\theta \in H^\infty(\mathbb{D}^n)$ be a one variable inner function and $n \geq 3$. Then $\mathcal{Q}_\theta$ is not essentially normal.*

Proof. Without loss of generality we may assume that $\theta(\boldsymbol{z}) = \theta'(z_1)$ for some inner function $\theta' \in H^\infty(\mathbb{D})$. Then it follows that $\mathcal{S}_\theta = \mathcal{S}_{\theta'} \otimes H^2(\mathbb{D}^{n-1})$ and

$$\mathcal{Q}_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n) = \mathcal{Q}_{\theta'} \otimes H^2(\mathbb{D}^{n-1}).$$

Now we compute the self commutator of C_{z_2}:

$$\begin{aligned}[C_{z_2}, C_{z_2}^*] &= P_{\mathcal{Q}_\theta} M_{z_2} M_{z_2}^*|_{\mathcal{Q}_\theta} - P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{Q}_\theta} M_{z_2}|_{\mathcal{Q}_\theta} \\ &= P_{\mathcal{Q}_\theta} M_{z_2} M_{z_2}^*|_{\mathcal{Q}_\theta} - I_{\mathcal{Q}_\theta} + P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta} M_{z_2}|_{\mathcal{Q}_\theta}.\end{aligned}$$

Using the fact

$$P_{\mathcal{S}_\theta} M_{z_2}|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = (P_{\mathcal{S}_{\theta'}} \otimes I_{H^2(\mathbb{D})} \otimes I_{H^2(\mathbb{D}^{n-2})}) M_{z_2}|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = 0,$$

and

$$M_{z_2}^*|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = 0,$$

we conclude that

$$[C_{z_2}, C_{z_2}^*]|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = -I_{\mathcal{Q}_\theta}|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = -I_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})}.$$

Since $n \geq 3$, $[C_{z_2}, C_{z_2}^*]|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})}$ is not compact, and hence the commutator $[C_{z_2}, C_{z_2}^*]$ is not compact. This completes the proof. $\square$

3. Essential normality

Our purpose in this section is to prove a list of results concerning essential normality for certain classes of quotient modules. We begin with the class of Beurling type quotient modules of $H^2(\mathbb{D}^n)$, $n \geq 3$.

Theorem 3.1. *Let θ be an inner function in $H^\infty(\mathbb{D}^n)$ and $n \geq 3$. Then $\mathcal{Q}_\theta$ is not essentially normal.*

Proof. First recall that the multiplication tuple $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$ is doubly commuting, that is

$$M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*$$

if $i \neq j$. Now by Lemma 2.5, we may assume without loss of generality that θ depends on both z_1 and z_2 variables. We show that $[C_{z_1}, C_{z_2}^*]$ is not compact. To this end, we compute

$$\begin{aligned}[C_{z_1}, C_{z_2}^*] &= P_{\mathcal{Q}_\theta} M_{z_1} M_{z_2}^*|_{\mathcal{Q}_\theta} - P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{Q}_\theta} M_{z_1}|_{\mathcal{Q}_\theta} = P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta} M_{z_1}|_{\mathcal{Q}_\theta} \\ &= P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta)} M_{z_1}|_{\mathcal{Q}_\theta} + P_{\mathcal{Q}_\theta} M_{z_2}^* P_{z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta} M_{z_1}|_{\mathcal{Q}_\theta}.\end{aligned}$$

Since M_{z_1} and M_{z_2} are isometries, we have

$$P_{\mathcal{Q}_\theta} M_{z_i}^* P_{z_i \mathcal{S}_\theta} = 0 \qquad (i = 1, 2).$$

This implies

$$P_{\mathcal{Q}_\theta} M_{z_2}^* P_{z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta} M_{z_1}|_{\mathcal{Q}_\theta} = 0,$$

and

$$[C_{z_1}, C_{z_2}^*] = P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta)} M_{z_1}|_{\mathcal{Q}_\theta}.$$

On the other hand, since $\mathcal{S}_\theta = \theta H^2(\mathbb{D}^n)$, we have

$$\begin{aligned}\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta) &= \theta H^2(\mathbb{D}^n) \ominus \theta(z_1 H^2(\mathbb{D}^n) + z_2 H^2(\mathbb{D}^n)) \\ &= \theta(\mathbb{C} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})).\end{aligned}$$

Then for $f \in \mathbb{C} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})$ and $g \in H^2(\mathbb{D}^n)$,

$$\langle M_{z_2}^* \theta f, \theta g \rangle = \langle f, z_2 g \rangle = 0,$$

and therefore

$$M_{z_2}^*(\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)) \subseteq \mathcal{Q}_\theta.$$

Consequently,

$$[C_{z_1}, C_{z_2}^*] = M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)} M_{z_1}|_{\mathcal{Q}_\theta}.$$

By Lemma 2.1, it is enough to show that $\langle [C_{z_1}, C_{z_2}^*]K_{\boldsymbol{w}}, K_{\boldsymbol{w}}\rangle$ does not converge to 0 as w_j approaches to $\partial\mathbb{D}$ for some fixed $3 \leq j \leq n$, and keeping all other co-ordinates of $\boldsymbol{w} = (w_1, \dots w_{j-1}, w_j, w_{j+1}, \dots, w_n) \in \mathbb{D}^n$ fixed. To this end, let $\boldsymbol{w} \in \mathbb{D}^n$. Since $\{\theta z_3^{m_3} \cdots z_n^{m_n} : m_3, \dots, m_n \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)$, we have

$$\begin{aligned}
P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)}(z_2K_{\boldsymbol{w}}) &= \sum_{m_3,\dots,m_n \in \mathbb{N}} \langle z_2K_{\boldsymbol{w}}, \theta z_3^{m_3} \cdots z_n^{m_n}\rangle \theta z_3^{m_3} \cdots z_n^{m_n} \\
&= \sum_{m_3,\dots,m_n \in \mathbb{N}} \langle K_{\boldsymbol{w}}, z_3^{m_3} \cdots z_n^{m_n}(M_{z_2}^*\theta)\rangle \theta z_3^{m_3} \cdots z_n^{m_n} \\
&= \frac{1}{\|\mathbb{S}(\cdot, \boldsymbol{w})\|}\theta \sum_{m_3,\dots,m_n \in \mathbb{N}} (\overline{w_3}z_3)^{m_3} \dots (\overline{w_n}z_n)^{m_n} \overline{M_{z_2}^*\theta(\boldsymbol{w})} \\
&= \overline{M_{z_2}^*\theta(\boldsymbol{w})} \prod_{j=1}^{2}(1 - |w_j|^2)^{\frac{1}{2}} \Big(\prod_{i=3}^{n} K_{w_i}\Big)\theta.
\end{aligned}$$

Here $K_{w_i} = K_{\boldsymbol{w}}$ with $\boldsymbol{w} = (0, \dots, w_i, \dots, 0)$. Thus

$$\begin{aligned}
\langle [C_{z_1}, C_{z_2}^*]K_{\boldsymbol{w}}, K_{\boldsymbol{w}}\rangle &= \langle M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)} M_{z_1} P_{\mathcal{Q}_\theta} K_{\boldsymbol{w}}, K_{\boldsymbol{w}}\rangle \\
&= \langle M_{z_1} P_{\mathcal{Q}_\theta} K_{\boldsymbol{w}}, P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)}(z_2K_{\boldsymbol{w}})\rangle \\
&= (M_{z_2}^*\theta)(\boldsymbol{w}) \prod_{j=1}^{2}(1 - |w_j|^2)^{\frac{1}{2}} \Big\langle M_{z_1} P_{\mathcal{Q}_\theta} K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i}\theta\Big\rangle \\
&= (M_{z_2}^*\theta)(\boldsymbol{w}) \prod_{j=1}^{2}(1 - |w_j|^2)^{\frac{1}{2}} \Big\langle M_{z_1}(1 - \overline{\theta(\boldsymbol{w})}\theta)K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i}\theta\Big\rangle,
\end{aligned}$$

where the last equality follows from (2.2). Since $M_{z_1}^*(\prod_{i=3}^{n} K_{w_i}) = 0$ and $M_\theta^* M_\theta = I_{H^2(\mathbb{D}^n)}$, we have

$$\Big\langle M_{z_1}\theta K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i}\theta\Big\rangle = \Big\langle \theta M_{z_1} K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i}\theta\Big\rangle = \Big\langle K_{\boldsymbol{w}}, M_{z_1}^*(\prod_{i=3}^{n} K_{w_i})\Big\rangle = 0.$$

Therefore,

$$\begin{aligned}
\langle [C_{z_1}, C_{z_2}^*] K_{\boldsymbol{w}}, K_{\boldsymbol{w}} \rangle &= (M_{z_2}^* \theta)(\boldsymbol{w}) \prod_{j=1}^{2} (1-|w_j|^2)^{\frac{1}{2}} \Big\langle M_{z_1} K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} \theta \Big\rangle \\
&= (M_{z_2}^* \theta)(\boldsymbol{w}) \prod_{j=1}^{2} (1-|w_j|^2)^{\frac{1}{2}} \Big\langle K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} (M_{z_1}^* \theta) \Big\rangle \\
&= (M_{z_2}^* \theta)(\boldsymbol{w}) \prod_{j=1}^{2} (1-|w_j|^2)^{\frac{1}{2}} \left(\overline{M_{z_1}^* \theta(\boldsymbol{w})} \frac{1}{\|\mathbb{S}(\cdot, \boldsymbol{w})\|} \prod_{i=3}^{n} \frac{1}{(1-|w_j|^2)^{\frac{1}{2}}} \right) \\
&= (M_{z_2}^* \theta)(\boldsymbol{w}) \, \overline{(M_{z_1}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1-|w_j|^2).
\end{aligned}$$

Since θ depends on both z_1 and z_2 variables, $M_{z_1}^* \theta$ and $M_{z_2}^* \theta$ are non-zero functions. Therefore it follows that there exist an $l \in \{3, \ldots, n\}$ and $\boldsymbol{w}_k = (w_1, \ldots, w_{l-1}, \lambda_k, w_{l+1}, \ldots, w_n) \in \mathbb{D}^n$ $(k \in \mathbb{N})$, where $\{\lambda_k\} \to \lambda \in \partial \mathbb{D}$ and w_i's are fixed, such that the limit of

$$(M_{z_2}^* \theta)(\boldsymbol{w}_k) \, \overline{(M_{z_1}^* \theta)(\boldsymbol{w}_k)} \prod_{j=1}^{2} (1-|w_j|^2)$$

as $k \to \infty$ is a non-zero number. This completes the proof. $\square$

We now proceed to the case of doubly commuting quotient modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Let $\mathcal{Q}$ be a quotient module of an analytic Hilbert module $\mathcal{H}_K$ over $\mathbb{C}[\boldsymbol{z}]$. It is known that $\mathcal{Q}$ is doubly commuting (that is, $[C_{z_i}, C_{z_j}^*] = 0$ for all $1 \leq i < j \leq n$) if and only if $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ for some quotient module $\mathcal{Q}_i$ of $\mathcal{H}_{k_i}$, $i = 1, \ldots, n$ (see [5], [17] and [19]).

THEOREM 3.2. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of an analytic Hilbert module $\mathcal{H}_K = \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}$ over $\mathbb{C}[\boldsymbol{z}]$, $n \geq 2$. Then $\mathcal{Q}$ is essentially normal if and only if one of the following holds:*

(i) *$\mathcal{Q}$ is finite dimensional.*
(ii) *There exits an $i \in \{1, \ldots, n\}$ such that $\mathcal{Q}_i$ is an infinite dimensional essentially normal quotient module of $\mathcal{H}_{k_i}$, and $\mathcal{Q}_j \cong \mathbb{C}$ for all $j \neq i$.*

Proof. Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be an infinite dimensional essentially normal quotient module. Then at least one of $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ is infinite dimensional. Without loss of generality we assume that $\mathcal{Q}_n$ is infinite dimensional. For each $i = 1, \ldots, n$, we now compute the self-commutator:

$$\begin{aligned}
[C_{z_i}, C_{z_i}^*] &= P_{\mathcal{Q}} M_{z_i} M_{z_i}^*|_{\mathcal{Q}} - P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} \\
&= P_{\mathcal{Q}_1} \otimes \cdots \otimes P_{\mathcal{Q}_{i-1}} \otimes \underbrace{[C_z, C_z^*]_i}_{\text{i-th place}} \otimes P_{\mathcal{Q}_{i+1}} \otimes \cdots \otimes P_{\mathcal{Q}_n},
\end{aligned} \tag{3.3}$$

where $[C_z, C_z^*]_i$ is the self-commutator corresponding to the quotient module $\mathcal{Q}_i$. Since $\mathcal{Q}_n$ is infinite dimensional, the compactness of $[C_{z_i}, C_{z_i}^*]$ implies that $[C_z, C_z^*]_i = 0$ for all $i = 1, \ldots, n-1$. Therefore, by Lemma 2.4, it follows that $\mathcal{Q}_i \cong \mathbb{C}$, $i = 1, \ldots, n-1$.

Finally, for $i = n$, the compactness of $[C_{z_n}, C_{z_n}^*] = P_{\mathcal{Q}_1} \otimes \cdots \otimes P_{\mathcal{Q}_{n-1}} \otimes [C_z, C_z^*]_n$ implies that $[C_z, C_z^*]_n$ is compact, that is, $\mathcal{Q}_n$ is essentially normal.

For the converse, it is enough to show that (ii) implies $\mathcal{Q}$ is essentially normal. Again, without loss of generality, we assume that $\mathcal{Q}_n$ is infinite dimensional essentially normal quotient module. Then it readily follows from (3.3) that $[C_{z_i}, C_{z_i}^*] = 0$, $i = 1, \ldots, n-1$, and $[C_{z_n}, C_{z_n}^*]$ is compact. Now the proof follows from Fuglede-Putnam theorem. $\square$

The above result applies, in particular, if $\mathcal{H}_K$ is $H^2(\mathbb{D}^n)$ or the weighted Bergman modules $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$ ($\boldsymbol{\alpha} \in \mathbb{Z}^n, \alpha_i > -1, i = 1, \ldots, n$). Moreover, since every quotient module of $H^2(\mathbb{D})$ is essentially normal, by Theorem 3.2 we have the following corollary.

Corollary 3.3. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, $n \geq 2$. Then $\mathcal{Q}$ is essentially normal if and only if one of the following holds:*

(i) *$\mathcal{Q}$ is finite dimensional.*
(ii) *There exits an $i \in \{1, \ldots, n\}$ such that $\mathcal{Q}_i$ is infinite dimensional, and $\mathcal{Q}_j \cong \mathbb{C}$ for all $j \neq i$.*

It is also well known that a quotient module $\mathcal{Q}$ of the Bergman module $L^2_a(\mathbb{D})$ is essentially normal if and only if

$$\dim(\mathcal{S} \ominus z\mathcal{S}) < \infty,$$

where $\mathcal{S} := L^2_a(\mathbb{D}) \ominus \mathcal{Q}$ is the corresponding submodule (see [22, Theorem 3.1]). Using this and Theorem 3.2, we have the following result.

Corollary 3.4. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of $L^2_a(\mathbb{D}^n)$, $n \geq 2$. Then $\mathcal{Q}$ is essentially normal if and only if one of the following holds:*

(i) *$\mathcal{Q}$ is finite dimensional.*
(ii) *There exists an $i \in \{1, \ldots, n\}$ such that $\mathcal{Q}_i$ is infinite dimensional with dim$(\mathcal{S}_i \ominus z\mathcal{S}_i) < \infty$ and $\mathcal{Q}_j \cong \mathbb{C}$ for all $j \neq i$, where $\mathcal{S}_i = L^2_a(\mathbb{D}) \ominus \mathcal{Q}_i$.*

We now restrict our attention to $H^2(\mathbb{D}^2)$, and formulate the definition of the Rudin quotient module of $H^2(\mathbb{D}^2)$ (see [6], [9]). Let $\Psi = \{\psi_n\}_{n=0}^\infty \subseteq H^2(\mathbb{D})$ be an increasing sequence of finite Blaschke products and $\Phi = \{\varphi_n\}_{n=0}^\infty \subseteq H^2(\mathbb{D})$ be a decreasing sequence of Blaschke products, that is, ψ_{n+1}/ψ_n and φ_n/φ_{n+1} are non-constant inner functions for all $n \in \mathbb{N}$. Then the *Rudin quotient module* corresponding to Ψ and Φ is denoted by $\mathcal{Q}_{\Psi,\Phi}$, and defined by

$$\mathcal{Q}_{\Psi,\Phi} := \bigvee_{n=0}^{\infty} \left(\mathcal{Q}_{\psi_n} \otimes \mathcal{Q}_{\varphi_n}\right).$$

We denote by $\mathcal{S}_{\Psi,\Phi}$ the submodule $H^2(\mathbb{D}^2) \ominus \mathcal{Q}_{\Psi,\Phi}$ corresponding to $\mathcal{Q}_{\Psi,\Phi}$. The following representations of $\mathcal{Q}_{\Psi,\Phi}$ and $\mathcal{S}_{\Psi,\Phi}$ are very useful:

$$\mathcal{Q}_{\Psi,\Phi} = \bigoplus_{n\geq 0} (\mathcal{Q}_{\psi_n} \ominus \mathcal{Q}_{\psi_{n-1}}) \otimes \mathcal{Q}_{\varphi_n} \quad \text{and} \quad \mathcal{S}_{\Psi,\Phi} = \mathcal{Q}' \otimes H^2(\mathbb{D}) \bigoplus_{n\geq 0} (\mathcal{Q}_{\psi_n} \ominus \mathcal{Q}_{\psi_{n-1}}) \otimes \mathcal{S}_{\varphi_n}, \tag{3.4}$$

where $\mathcal{Q}_{\psi_{-1}} := \{0\}$ and $\mathcal{Q}' = H^2(\mathbb{D}) \ominus \vee_{n\geq 0} \mathcal{Q}_{\psi_n}$. The first equality follows from the fact that $\mathcal{Q}_{\psi_n} \subseteq \mathcal{Q}_{\psi_{n+1}}$ and $\mathcal{Q}_{\varphi_n} \supseteq \mathcal{Q}_{\varphi_{n+1}}$ $(n \geq 0)$ and the second equality can be checked easily using the equality $\mathcal{Q}_{\Psi,\Phi} \oplus \mathcal{S}_{\Psi,\Phi} = H^2(\mathbb{D}^2)$.

Next we show that the Rudin quotient modules are not essentially normal.

THEOREM 3.5. *Let $\mathcal{Q}_{\Psi,\Phi}$ be a Rudin quotient module of $H^2(\mathbb{D}^2)$ corresponding to an increasing sequence of finite Blaschke products $\Psi = \{\psi_n\}_{n\geq 0}$ and a decreasing sequence of Blaschke products $\Phi = \{\varphi_n\}_{n\geq 0}$. Then $\mathcal{Q}_{\Psi,\Phi}$ is not essentially normal.*

Proof. Let b_β, the Blaschke factor corresponding to $\beta \in \mathbb{D}$, be a factor of ψ_{m+1}/ψ_m for some $m \geq 0$. For contradiction, we assume that $\mathcal{Q}_{\Psi,\Phi}$ is essentially normal. Since $\mathcal{Q} := \mathcal{Q}_{\Psi,\Phi}$ is essentially normal, for a polynomial p it is easy to verify using a simple commutator manipulation that $[C_{p(z_1)}, C^*_{p(z_1)}]$ is compact, where $C_{p(z_1)} = P_{\mathcal{Q}} M_{p(z_1)}|_{\mathcal{Q}}$. Now as ψ_m is a finite Blaschke product and can be approximated by polynomials, $[C_{\psi_m(z_1)}, C^*_{\psi_m(z_1)}]$ is also compact, where $C_{\psi_m(z_1)} = P_{\mathcal{Q}} M_{\psi_m(z_1)}|_{\mathcal{Q}}$. Now setting $\mathcal{S} := \mathcal{S}_{\Psi,\Phi}$, we have

$$\begin{aligned}
[C_{\psi_m(z_1)}, C^*_{\psi_m(z_1)}] &= P_{\mathcal{Q}} M_{\psi_m(z_1)} M^*_{\psi_m(z_1)}|_{\mathcal{Q}} - P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{Q}} M_{\psi_m(z_1)}|_{\mathcal{Q}} \\
&= -P_{\mathcal{Q}}(I - M_{\psi_m(z_1)} M^*_{\psi_m(z_1)})|_{\mathcal{Q}} + P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{S}} M_{\psi_m(z_1)}|_{\mathcal{Q}} \\
&= -P_{\mathcal{Q}}(P_{\mathcal{Q}_{\psi_m}} \otimes I)|_{\mathcal{Q}} + P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{S}} M_{\psi_m(z_1)}|_{\mathcal{Q}}. \qquad (3.5)
\end{aligned}$$

Since φ_{m+1} is an infinite Blaschke product, there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in the zero set of φ_{m+1} such that $K_{\lambda_i} \in \mathcal{Q}_{\varphi_{m+1}}$ and λ_i approaches to $\partial\mathbb{D}$ as $i \to \infty$. Furthermore, since $K_\beta \otimes K_{\lambda_i} \in \mathcal{Q}_{\psi_{m+1}} \otimes \mathcal{Q}_{\varphi_{m+1}} \subseteq \mathcal{Q}$ and $\psi_m K_\beta \otimes K_{\lambda_i} \in (\mathcal{Q}_{\psi_{m+1}} \ominus \mathcal{Q}_{\psi_m}) \otimes \mathcal{Q}_{\varphi_{m+1}}$ by the divisibility of ψ_{m+1} and ψ_m, we have $P_{\mathcal{S}}(\psi_m K_\beta \otimes K_{\lambda_i}) = 0$, $i \in \mathbb{N}$ (by (3.4)). Thus

$$P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{S}} M_{\psi_m(z_1)}(K_\beta \otimes K_{\lambda_i}) = 0 \quad (i \in \mathbb{N}).$$

Finally, from (3.5), we have

$$\begin{aligned}
\langle [C_{\psi_m(z_1)}, C^*_{\psi_m(z_1)}](K_\beta \otimes K_{\lambda_i}), K_\beta \otimes K_{\lambda_i} \rangle &= -\langle (P_{\mathcal{Q}_{\psi_m}} K_\beta) \otimes K_{\lambda_i}, K_\beta \otimes K_{\lambda_i} \rangle \\
&= -\langle (1 - \overline{\psi_m(\beta)}\psi_m) K_\beta, K_\beta \rangle \\
&= -(1 - |\psi_m(\beta)|^2),
\end{aligned}$$

which does not converges to 0 as λ_i approaches to $\partial\mathbb{D}$. Thus by Lemma 2.1, we have the desired contradiction. This completes the proof. ☐

REMARK 3.6. *Let $m > 1$. For a decreasing sequence of Blaschke products $\{\varphi_n\}_{n=1}^m$ and an increasing sequence of finite Blaschke products $\{\psi_n\}_{n=1}^m$, we consider the quotient module*

$$\mathcal{Q} = \bigvee_{n=1}^{m} \mathcal{Q}_{\psi_n} \otimes \mathcal{Q}_{\varphi_n}.$$

Adapting the techniques in the proof of the above theorem, one can conclude that $\mathcal{Q}$ is essentially normal if and only if φ_n is a finite Blaschke product for all $n = 1, \ldots, m$. In other words, $\mathcal{Q}$ is essentially normal if and only if $\mathcal{Q}$ is finite dimensional.

4. Boundary Representations for doubly commuting quotient modules

In this section we study boundary representations for doubly commuting quotient modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. First, we prove a general result in the setting of minimal tensor products of C^*-algebras. Before that we fix some notations. We denote by

$V_1 \underline{\otimes} V_2$ the algebraic tensor product of two vector spaces V_1 and V_2, and by $A_1 \otimes A_2$ the minimal tensor product of two C^*-algebras A_1 and A_2 where the norm on $A_1 \otimes A_2$ is obtained via the identification $A_1 \otimes A_2 \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ corresponding to any faithful representations of A_1 and A_2 in $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively.

The base case ($n = 2$) of the following result is due to Hopenwasser (see Lemmas 1 and 3 in [16]). The proof for the general case n can be obtain easily by applying Hopenwasser's result $n - 1$ times and therefore we omit the proof.

LEMMA 4.1 (cf. [16]). *Let A_i be a unital subalgebra of $\mathcal{B}(\mathcal{H}_i)$ for some Hilbert space $\mathcal{H}_i$, and let $C^*(A_i)$ be the irreducible C^*-algebra generated by A_i in $\mathcal{B}(\mathcal{H}_i)$, $i = 1, \ldots, n$. Set $A := \overline{(A_1 \underline{\otimes} \cdots \underline{\otimes} A_n)}$, the norm closure of $A_1 \underline{\otimes} \cdots \underline{\otimes} A_n$ in $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$. Then the following are equivalent.*

(i) *The identity representation of $C^*(A_1) \otimes \cdots \otimes C^*(A_n)$ is a boundary representation relative to A.*
(ii) *The identity representation of $C^*(A_i)$ is a boundary representation relative to A_i for all $i = 1, \ldots, n$.*

As a straightforward consequence of the above lemma we obtain the following:

THEOREM 4.2. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}$ over $\mathbb{C}[\boldsymbol{z}]$. Then the following are equivalent.*

(i) *The identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$.*
(ii) *The identity representation of $C^*(\mathcal{Q}_i)$ is a boundary representation relative to $B(\mathcal{Q}_i)$ for all $i = 1, \ldots, n$.*

Proof. The result follows from Lemma 4.1 and the fact that

$$C^*(\mathcal{Q}) = C^*(\mathcal{Q}_1) \otimes \cdots \otimes C^*(\mathcal{Q}_n),$$

and

$$B(\mathcal{Q}) = \overline{B(\mathcal{Q}_1) \underline{\otimes} \cdots \underline{\otimes} B(\mathcal{Q}_n)},$$

where the closure is in the norm topology of $B(\mathcal{Q})$. ☐

The following result is now an immediate consequence of Theorems 1.2 and 4.2.

COROLLARY 4.3. *Let $\mathcal{Q} = \mathcal{Q}_{\theta_1} \otimes \cdots \otimes \mathcal{Q}_{\theta_n}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, where θ_i, $i = 1, \ldots, n$, is a one variable inner function. Then the following are equivalent.*

(i) *The identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$.*
(ii) *The identity representation of $C^*(\mathcal{Q}_{\theta_i})$ is a boundary representation relative to $B(\mathcal{Q}_{\theta_i})$ for all $i = 1, \ldots, n$.*
(iii) *For all $i = 1, \ldots, n$, Z_{θ_i} is a proper subset of $\mathbb{T}$, where Z_{θ_i} consists of all points λ on $\mathbb{T}$ for which θ_i cannot be continued analytically from $\mathbb{D}$ to λ.*

Now we turn to the case of the Bergman module $L^2_a(\mathbb{D}^n)$. For $n = 1$, boundary representations corresponding to a quotient module of $L^2_a(\mathbb{D})$ are studied in [15]. For a submodule $\mathcal{S}$ of $L^2_a(\mathbb{D})$, set

$$Z_*(\mathcal{S}) := \bigcap_{f \in \mathcal{S}} Z_*(f),$$

where

$$Z_*(f) = \big\{\lambda \in \mathbb{D} : f(\lambda) = 0\big\} \cup \big\{\lambda \in \mathbb{T} : \liminf_{z\in\mathbb{D}, z\to\lambda} |f(z)| = 0\big\}.$$

It is easy to see that for a finite dimensional quotient module $\mathcal{Q}$ of $L^2_a(\mathbb{D})$, the identity representation of $C^*(\mathcal{Q})$ is always a boundary representation relative to $B(\mathcal{Q})$. On the other hand, for an infinite dimensional $\mathcal{Q}$, the identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$ if and only if $\dim(\mathcal{S} \ominus z\mathcal{S}) = 1$ and $Z_*(\mathcal{S})$ is a proper subset of $\mathbb{T}$, where $\mathcal{S} = L^2_a(\mathbb{D}) \ominus \mathcal{Q}$ is the corresponding submodule (see [15, Theorem 1.2]). Using this and Theorem 4.2, we have the following result.

Corollary 4.4. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of $L^2_a(\mathbb{D}^n)$. Then the following are equivalent.*

(i) *The identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$.*
(ii) *The identity representation of $C^*(\mathcal{Q}_i)$ is a boundary representation relative to $B(\mathcal{Q}_i)$ for all $i = 1, \ldots, n$.*
(iii) *If $\mathcal{Q}_i$ $(1 \le i \le n)$ is infinite dimensional then dim$(\mathcal{S}_i \ominus z\mathcal{S}_i) = 1$ and $Z_*(\mathcal{S}_i)$ is a proper subset of $\mathbb{T}$, where $\mathcal{S}_i = L^2_a(\mathbb{D}) \ominus \mathcal{Q}_i$ is the corresponding submodule.*

5. Boundary representations for homogeneous quotient modules

The purpose of this section is to investigate boundary representations for homogeneous quotient modules of $H^2(\mathbb{D}^2)$. We begin with a lemma which is a standard application of Arveson's theory on boundary representations [1, 2]. For generality, we prove it for quotient modules of $H^2(\mathbb{D}^n)$. For an essentially normal quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$, we denote by $\sigma_e(\mathcal{Q})$ the essential joint spectrum of $(C_{z_1}, \ldots, C_{z_n})$.

Lemma 5.1. *Let $\mathcal{Q}$ be an essentially normal quotient module of $H^2(\mathbb{D}^n)$.*

(a) *If there exists a matrix-valued polynomial p such that*

$$\|p(C_{z_1}, \ldots, C_{z_n})\| > \|p\|^\infty_{\sigma_e(\mathcal{Q})} := \sup_{\boldsymbol{z}\in\sigma_e(\mathcal{Q})} \|p(\boldsymbol{z})\|,$$

then the identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $\mathcal{B}(\mathcal{Q})$.

(b) *If the commuting tuple $(C_{z_1}, \ldots, C_{z_n})$ has a normal dilation on $\sigma_e(\mathcal{Q})$, then the identity representation of $C^*(\mathcal{Q})$ is not a boundary representation relative to $\mathcal{B}(\mathcal{Q})$.*

Proof. a) Since $\mathcal{Q}$ is essentially normal and $C^*(\mathcal{Q})$ is irreducible, we have that $K(\mathcal{Q}) \subseteq C^*(\mathcal{Q})$ and the following extension

$$0 \longrightarrow K(\mathcal{Q}) \hookrightarrow C^*(\mathcal{Q}) \longrightarrow C(\sigma_e(\mathcal{Q})) \longrightarrow 0.$$

If there exists a matrix-valued polynomial p such that

$$\|p(C_{z_1}, \ldots, C_{z_n})\| > \|p\|^\infty_{\sigma_e(\mathcal{Q})},$$

then the restriction of the canonical contractive homomorphism

$$q : C^*(\mathcal{Q}) \to C^*(\mathcal{Q})/K(\mathcal{Q}) \cong C(\sigma_e(\mathcal{Q}))$$

to $\mathcal{B}(\mathcal{Q})$ is not a complete isometry. The desired conclusion now follows from the Arveson's boundary theorem [2, Theorem 2.1.1].

(b) The existence of a normal dilation implies that the above completely contractive map q restricted to the linear span of $\mathcal{B}(\mathcal{Q}) \cup \mathcal{B}(\mathcal{Q})^*$ is a complete isometry. Then the conclusion again follows from the Arveson's boundary theorem [2, Theorem 2.1.1]. $\square$

The following basic property of two variable homogeneous polynomials will be used subsequently. Given a homogeneous polynomial $p \in \mathbb{C}[z_1, z_2]$ there exist homogeneous polynomials $p_1, p_2 \in \mathbb{C}[z_1, z_2]$, unique up to a scalar multiple of modulus one, such that

$$p = p_1 p_2,$$

and

$$Z(p_1) \cap \partial\mathbb{D}^2 \subset \mathbb{T}^2 \quad \text{and} \quad Z(p_2) \cap \partial\mathbb{D}^2 \subset (\mathbb{D} \times \mathbb{T}) \cup (\mathbb{T} \times \mathbb{D}).$$

Let $pH^2(\mathbb{D}^2)$ denote the submodule of $H^2(\mathbb{D}^2)$ generated by p. Suppose that $\mathcal{Q}_p$ is the corresponding quotient module of $H^2(\mathbb{D}^2)$, that is,

$$\mathcal{Q}_p = H^2(\mathbb{D}^2) \ominus pH^2(\mathbb{D}^2).$$

The following characterization of essential normality of $\mathcal{Q}_p$ is due to Guo and Wang [14, Theorem 1.1].

Theorem 5.2 (Guo & Wang, [14]). *Let p be a non-zero homogeneous polynomial in $\mathbb{C}[z_1, z_2]$, and $p = p_1 p_2$ be the factorization of p as above. Then the quotient module $\mathcal{Q}_p$ is essentially normal if and only if p_2 has one of the following forms:*

(i) $p_2 \equiv c$ *with* $c \neq 0$,
(ii) $p_2 = \alpha z_1 + \beta z_2$ *with* $|\alpha| \neq |\beta|$,
(iii) $p_2 = c(z_1 - \alpha z_2)(z_2 - \beta z_1)$ *with* $|\alpha| < 1, |\beta| < 1$ *and* $c \neq 0$.

The following result in [14] gives a description of the essential joint spectrum of the above type of quotient modules. For a proof we refer the reader to [14, Theorem 6.2].

Lemma 5.3 (Guo & Wang, [14]). *Let p be a homogeneous polynomial. Then*

$$\sigma_e(\mathcal{Q}_p) = Z(p) \cap \partial\mathbb{D}^2.$$

For our present purposes, however, we need only the fact that $\sigma_e(\mathcal{Q}_p) \subset Z(p) \cap \partial\mathbb{D}^2$.

We now state our main result of this section. This gives a partial characterization of boundary representations for the class of essentially normal homogeneous quotient modules of $H^2(\mathbb{D}^2)$.

Theorem 5.4. *Let $p \in \mathbb{C}[z_1, z_2]$ be a homogeneous polynomial. Suppose that $\mathcal{Q}_p$ is an essentially normal quotient module of $H^2(\mathbb{D}^2)$. Then the identity representation of $C^*(\mathcal{Q}_p)$ is a boundary representation relative to $B(\mathcal{Q}_p)$ if p is not of the following form:*
(i) $p = c(z_1^m - \alpha z_2^m)$ *for some* $m \in \mathbb{N}$, $c \neq 0$ *and* $|\alpha| = 1$,
(ii) $p = \alpha z_1 + \beta z_2$ *with* $|\alpha| \neq |\beta|$.

Furthermore, if p is either as in (i) *with $m = 1$ or as in* (ii) *then the identity representation of $C^*(\mathcal{Q}_p)$ is not a boundary representation.*

Our proof of Theorem 5.4 on boundary representations for homogeneous quotient modules is based on the following two special cases. A couple of lemmas below describes these.

LEMMA 5.5. *Let $p = c\prod_{i=1}^{m}(z_1 - \alpha_i z_2)^{n_i}$ be a homogeneous polynomial with $c \neq 0$ and α_i's are distinct scalars of modulus one. Assume further that $n_i > 1$ for some $i = 1, \ldots, m$. Then the identity representation of $C^*(\mathcal{Q}_p)$ is a boundary representation relative to $B(\mathcal{Q}_p)$.*

Proof. Without loss of any generality assume that $n_1 > 1$. Set

$$q(z_1, z_2) := (z_1 - \alpha_1 z_2)\prod_{i=2}^{m}(z_1 - \alpha_i z_2)^{n_i}.$$

Then, $q(C_{z_1}, C_{z_2})$ is a non-zero operator and $\|q\|^{\infty}_{Z(p)} = 0$. Hence, by Lemma 5.3 and part (a) of Lemma 5.1, the identity representation of $C^*(\mathcal{Q}_p)$ is a boundary representation relative to $B(\mathcal{Q}_p)$. □

LEMMA 5.6. *Let $p = c(z_1 - \alpha z_2)$, for some $\alpha \in \mathbb{C}$ and $c \neq 0$. Then, the identity representation of $C^*(\mathcal{Q}_p)$ is not a boundary representation relative to $B(\mathcal{Q}_p)$.*

Proof. Let $\alpha = 0$. Then $\mathcal{Q}_p$ is unitarily equivalent to $H^2(\mathbb{D})$ and, hence, the conclusion follows easily. Now let $|\alpha| = 1$. In this case $\mathcal{Q}_p$ is unitary equivalent to the Bergman space over the unit disc and therefore the result follows. Finally, let $\alpha \neq 0$ and $|\alpha| \neq 1$. Then

$$\frac{1}{\alpha}C_{z_1} = C_{z_2},$$

and hence $C^*(\mathcal{Q}_p)$ is generated by C_{z_1}. Now assume that $|\alpha| > 1$ (the $|\alpha| < 1$ case is similar). Set $\beta = \frac{\alpha}{|\alpha|^2}$ and

$$c_n := (\sum_{m=0}^{n} |\beta|^{2m})^{1/2} \qquad (n \in \mathbb{N}).$$

It follows that the sequence of homogeneous polynomials $\{p_n\}$ is an orthonormal basis for $\mathcal{Q}_p$ [11], where

$$p_n(z) = \frac{1}{c_n}\sum_{m=0}^{n} z_1^m(\beta z_2)^{n-m} \quad (n \in \mathbb{N}).$$

Moreover (again see [11]), for all $n \geq 0$,

$$\begin{aligned} C_{z_1}(p_n) &= P_{\mathcal{Q}_p}(\frac{1}{c_n}\sum_{m=0}^{n} z_1^{m+1}(\beta z_2)^{n-m}) \\ &= \frac{1}{c_n}\langle p_{n+1}, \sum_{m=0}^{n} z_1^{m+1}(\beta z_2)^{n-m}\rangle\ p_{n+1} \\ &= \frac{c_n}{c_{n+1}}p_{n+1}. \end{aligned}$$

Thus, by Theorem 5.2, C_{z_1} is an essentially normal weighted shift with weights $\{\frac{c_n}{c_{n+1}}\}_{n \geq 0}$. Finally, since $\limsup_n \frac{c_n}{c_{n+1}} = \sup_n \frac{c_n}{c_{n+1}}$, the result follows from [2, Corollary 2]. □

We are now in a position to give a proof of Theorem 5.4.

Proof of Theorem 5.4. We first note that, since $\mathcal{Q}_p$ is essentially normal, p can be represented as in Theorem 5.2. Now by Lemma 5.5, is it enough to consider the case $p = p_1 p_2$, where

$$p_1(z) = \prod_{i=1}^{m} (z_1 - \alpha_i z_2),$$

$\alpha_i,\ i = 1, \dots, m$, are all distinct scalars of modulus one, and p_2 is as in Theorem 5.2. In view of the forms of p_2 in Theorem 5.2, we next consider the following four cases.

Case I: Let $p_2 = c$, $p_1 = \prod_{i=1}^{m} (z_1 - \alpha_i z_2)$, $\alpha_i,\ i = 1, \dots, m$, are all distinct scalars of modulus one, and that $p = p_1 p_2$ is not of the form $c(z_1^m - \alpha z_2^m)$.
In this case we have $m > 1$. Set

$$q(z) = \frac{1}{z_1} \Big(\prod_{i=1}^{m} (z_1 - \alpha_i z_2) - (-1)^m \big(\prod_{i=1}^{m} \alpha_i \big) z_2^m \Big).$$

Then

$$q(z) = \sum_{k=0}^{m-1} (-1)^k \left(\sum_{1 \le i_1 < \dots < i_k \le m} \alpha_{i_1} \dots \alpha_{i_k} \right) z_1^{m-k-1} z_2^k. \tag{5.6}$$

A simple calculation shows that

$$\|q\|^{\infty}_{Z(p) \cap \partial \mathbb{D}^2} = 1.$$

On the other hand, note that $1 \in \mathcal{Q}_p$ as p vanishes at 0 and $q \in \mathcal{Q}_p$ as the degree of q is strictly less than that of the homogeneous polynomial p. Then we have

$$\|q(C_{z_1}, C_{z_2})\| \ge \|q\|_{H^2(\mathbb{D}^2)} = \sqrt{1 + \sum_{k=1}^{m-1} \left| \sum_{1 \le i_1 < \dots < i_k \le m} \alpha_{i_1} \dots \alpha_{i_k} \right|^2}.$$

Now if

$$\sum_{k=1}^{m-1} \left| \sum_{1 \le i_1 < \dots < i_k \le m} \alpha_{i_1} \dots \alpha_{i_k} \right|^2 = 0,$$

then (5.6) yields that

$$p_1(z) = \prod_{i=1}^{m} (z_1 - \alpha_i z_2) = z_1^m - \alpha z_2^m,$$

for some α of modulus one which is an obvious contradiction. Thus,

$$\|q(C_{z_1}, C_{z_2})\| > 1,$$

and therefore, by Lemma 5.3 and Lemma 5.1, the identity representation is a boundary representation in this case.

Case II: Let $p_2(z) = (z_1 - \gamma z_2)$, $|\gamma| \neq 1$, $p_1(z) = \prod_{i=1}^{m} (z_1 - \alpha_i z_2)$, where $\alpha_i,\ i = 1, \dots, m$, are distinct scalars of modulus one. Also, without loss of generality, we may assume that $|\gamma| < 1$.

Otherwise, by interchanging the role of z_1 and z_2, we can consider that $p_2(z) = (z_2 - \frac{1}{\gamma} z_1)$ and $p_1(z) = \prod_{i=1}^{m} (z_2 - \frac{1}{\alpha_i} z_1)$. As in the previous case, set

$$q(z) = \frac{1}{z_1} \Big(\prod_{i=1}^{m+1} (z_1 - \alpha_i z_2) - (-1)^{m+1} \prod_{i=1}^{m+1} \alpha_i z_2^{m+1} \Big),$$

where $\alpha_{m+1} = \gamma$. A similar computation shows that

$$\|q\|^{\infty}_{Z(p) \cap \partial \mathbb{D}^2} = 1,$$

and

$$\|q(C_{z_1}, C_{z_2})\| \geq \sqrt{1 + \sum_{k=1}^{m} \left| \sum_{1 \leq i_1 < \cdots < i_k \leq m+1} \alpha_{i_1} \ldots \alpha_{i_k} \right|^2 }.$$

Therefore, if

$$\sum_{k=1}^{m} \left| \sum_{1 \leq i_1 < \cdots < i_k \leq m+1} \alpha_{i_1} \ldots \alpha_{i_k} \right|^2 = 0,$$

then

$$\prod_{i=1}^{m+1} (z_1 - \alpha_i z_2) = z_1^{m+1} - \alpha z_2^{m+1},$$

for some scalar α with $|\alpha| \neq 1$. Consequently

$$|\alpha_1| = \cdots = |\alpha_{m+1}| = |\alpha|^{1/(m+1)} \neq 1,$$

which is a contradiction. Therefore $\|q(C_{z_1}, C_{z_2})\| > 1$, and the conclusion again follows from Lemma 5.3 and Lemma 5.1.

Case III: Let $p = p_2 = (z_1 - \gamma_1 z_2)(z_2 - \gamma_2 z_1)$, $|\gamma_1| < 1$ and $|\gamma_2| < 1$. We further divide it into two sub-cases. First assume that $\gamma_2 \neq 0$. In this sub-case, without any loss of generality, we take $p = (z_1 - \gamma_1 z_2)(z_1 - \gamma_2 z_2)$ with $|\gamma_1| < 1$ and $|\gamma_2| > 1$. For each $\epsilon > 0$, set

$$q_\epsilon := (z_1 - \epsilon \gamma_2 z_2) \in \mathbb{C}[z_1, z_2],$$

and set

$$V_1 = \{(\gamma_1 z_2, z_2) \in \partial \mathbb{D}^2 : |z_2| = 1\} \text{ and } V_2 = \{(\gamma_2 z_2, z_2) \in \partial \mathbb{D}^2 : |\gamma_2 z_2| = 1\}.$$

Note that

$$Z(p) \cap \partial \mathbb{D}^2 = V_1 \cup V_2,$$

and

$$\|q_\epsilon\|^{\infty}_{V_1} = |\gamma_1 - \epsilon \gamma_2|, \quad \|q_\epsilon\|^{\infty}_{V_2} = (1 - \epsilon).$$

Therefore, for a sufficiently small $0 < \epsilon < 1$, we obtain

$$\|q_\epsilon\|^{\infty}_{Z(p) \cap \partial \mathbb{D}^2} < 1.$$

On the other hand, for any $0 < \epsilon < 1$,

$$\|q_\epsilon(C_{z_1}, C_{z_2})\| \geq \sqrt{1 + |\epsilon \gamma_2|^2} > 1,$$

and the conclusion again follows from Lemma 5.1.

If $\gamma_2 = 0$, then $p = z_2(z_1 - \gamma_1 z_2)$. For each $\epsilon > 0$, consider

$$q_\epsilon = z_1 - \epsilon z_2,$$

and set

$$V_1 = \{(\gamma_1 z_2, z_2) \in \partial\mathbb{D}^2 : |z_2| = 1\} \text{ and } V_2 = \{(z_1, 0) \in \partial\mathbb{D}^2 : |z_1| = 1\}.$$

Then as before, one can check that

$$Z(p) \cap \partial\mathbb{D}^2 = V_1 \cup V_2,\ \|q_\epsilon\|^\infty_{V_1} = |\gamma_1 - \epsilon| \ \text{ and } \ \|q_\epsilon\|^\infty_{V_2} = 1.$$

Thus, for a sufficiently small $0 < \epsilon < 1$, we have

$$\|q_\epsilon\|^\infty_{Z(p)\cap\partial\mathbb{D}^2} = 1.$$

On the other hand, for any $0 < \epsilon < 1$,

$$\|q_\epsilon(C_{z_1}, C_{z_2})\| \geq \sqrt{1 + |\epsilon|^2} > 1,$$

and therefore the conclusion follows from Lemma 5.1.

Case IV: Let $p_2(z) = (z_1 - \gamma_1 z_2)(z_2 - \gamma_2 z_1)$, $|\gamma_1| < 1$, $|\gamma_2| < 1$, $p_1(z) = \prod_{i=1}^m (z_1 - \alpha_i z_2)$, and $\alpha_i,\ i = 1, \ldots, m$, are distinct scalars of modulus one. Set

$$V_{\gamma_1} = \{(\gamma_1 z_2, z_2) \in \partial\mathbb{D}^2 : |z_2| = 1\}, \quad V_{\gamma_2} = \{(z_1, \gamma_2 z_1) \in \partial\mathbb{D}^2 : |z_1| = 1\},$$

and

$$V_{\alpha_i} := \{(\alpha_i z_2, z_2) \in \partial\mathbb{D}^2 : |z_2| = 1\}, \quad i = 1, \ldots, m.$$

We now consider, for $\epsilon > 0$,

$$q_\epsilon = z_2(z_1^m + \epsilon q') \in \mathbb{C}[z_1, z_2],$$

where

$$q'(z) = \prod_{i=1}^m (z_1 - \alpha_i z_2) - z_1^m = \sum_{k=1}^m (-1)^k \left(\sum_{1 \leq i_1 < \cdots < i_k \leq m} \alpha_{i_1} \ldots \alpha_{i_k} \right) z_1^{m-k} z_2^k.$$

Then again by a simple calculation we get

$$\|q_\epsilon\|^\infty_{V_{\gamma_1}} \leq |\gamma_1^m| + \epsilon M, \quad \|q_\epsilon\|^\infty_{V_{\gamma_2}} \leq |\gamma_2|(1 + \epsilon M),$$

and

$$\|q_\epsilon\|^\infty_{V_{\alpha_i}} = |(1 - \epsilon)|, \quad i = 1, \ldots, m,$$

where

$$M = \max\{|q'(\gamma_1, 1)|, |q'(1, \gamma_2)|\}.$$

We now choose $0 < \epsilon < 1$ so that $\|q_\epsilon\|^\infty_{Z(p)\cap\partial\mathbb{D}^2} < 1$. On the other hand, since $\|q_\epsilon(C_{z_1}, C_{z_2})\| \geq \|q_\epsilon\| > 1$, the conclusion follows immediately. Thus we have the first part of the theorem.

The last part follows from Lemma 5.6. This completes the proof of the theorem. $\square$

Note that Theorem 5.4 completely describes the issue of boundary representations for all essentially normal homogeneous quotient modules of $H^2(\mathbb{D}^2)$, except when $p = (z_1^m - \alpha z_2^m)$, $|\alpha| = 1$ and $m \geq 2$. We conclude this paper with the following question: Let $|\alpha| = 1$, $m \geq 2$ and let $p = z_1^m - \alpha z_2^m$. Is the identity representation of $C^*(\mathcal{Q}_p)$ a boundary representation?

Acknowledgment: We are very grateful to the referee for a careful reading of the manuscript and valuable suggestions and thoughtful comments. The first two authors are grateful to Indian Statistical Institute, Bangalore Center for warm hospitality. The first named author's research work is supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094. The second named author is supported by an INSPIRE faculty fellowship (IFA-MA-02) funded by DST. The third author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014.

References

[1] W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141-224.
[2] W. Arveson, *Subalgebras of C^*-algebras II*, Acta Math. **128** (1972), 271-308.
[3] W. Arveson, *Quotients of standard Hilbert modules*, Trans. Amer. Math. Soc. **359** (2007), 6027-6055.
[4] L. Brown, R. Douglas and P. Fillmore, *Extension of C^*-algebras and K-homology*, Ann. of Math. **105** (1977), 265-324.
[5] A. Chattopadhyay, B. K. Das and J. Sarkar, *Tensor product of quotient Hilbert modules*, J. Math. Anal. Appl. **429** (2015), 727-747.
[6] A. Chattopadhyay, B. K. Das and J. Sarkar, *Star-generating vectors of Rudin's quotient modules*, J. Funct. Anal. **267** (2014), 4341-4360.
[7] X. Chen and K. Guo, *Analytic Hilbert modules*, π-Chapman & Hall/CRC Res. Notes Math. 433, 2003.
[8] D. Clark, *Restrictions of H^p functions in the polydisk*, Amer. J. math. **110** (1988), 1119-1152.
[9] B. K. Das and J. Sarkar, *Rudin's Submodules of $H^2(\mathbb{D}^2)$*, C. R. Acad. Sci. Paris **353** (2015), 51-55.
[10] K. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke Math. J. **164** (2015), 2989-3004.
[11] R. G. Douglas and V. I. Paulsen, *Hilbert Modules over Function Algebras*, Research Notes in Mathematics Series, 47, Longman, Harlow, 1989.
[12] K. Guo and K. Wang, *Essentially normal Hilbert modules and K-homology*, Math. Ann. **340** (2008), 907-934.
[13] K. Guo and K. Wang, *Beurling type quotient modules over the bidisk and boundary representations*, J. Funct. Anal. **257** (2009), 3218-3238.
[14] K. Guo and P. Wang, *Essentially normal Hilbert modules and K-homology III: Homogeneous quotient modules on the bidisk*, Sci. China Ser. A **50** (2007), 387-411.
[15] W. He, *Boundary representations on co-invariant subspaces of Bergman space*, Proc. Amer. Math. Soc. **138** (2010), 615-622.
[16] A. Hopenwasser, *Boundary representations and tensor products of C^*-algebras*, Proc. Amer. Math. Soc. **71** (1978), 95-98.
[17] K. Izuchi, T. Nakazi and M. Seto, *Backward shift invariant subspaces in the bidisc II*, J. Oper. Theory **51** (2004), 361-376.
[18] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.
[19] J. Sarkar, *Jordan blocks of $H^2(\mathbb{D}^n)$*, J. Oper. Theory **72** (2014), 371-385.
[20] J. Sarkar, A. Sasane and B. Wick, *Doubly commuting submodules of the Hardy module over polydiscs*, Studia Mathematica **217** (2013), 179-192.
[21] P. Wang, *The essential normality of $N_{\underline{\eta}}$-type quotient module of Hardy module on the polydisc*, Proc. Amer. Math. Soc. **142** (2014), 151-156.

[22] K. Zhu, *Restriction of the Bergman shift to an invariant subspace*, Quart. J. Math. Oxford Ser. (2) **48** (1997), 519-532.

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, India- 400076

E-mail address: dasb@math.iitb.ac.in, bata436@gmail.com

Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur 741 246, West Bengal, India

E-mail address: sushil.gorai@iiserkol.ac.in

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India

E-mail address: jay@isibang.ac.in, jaydeb@gmail.com